ГЕНЕРАЦИЯ СИСТЕМ УРАВНЕНИЙ ПРИ РЕШЕНИИ ОДНОМЕРНЫХ ЗАДАЧ ДИРИХЛЕ И НЕЙМАНА МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ

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?????????? ??????. ?????????? ??????? ??????

$$-\frac{d^2u}{dx^2}+u=f$$

??? $\forall u \in C_2[0;1]$, (1)

??????? ???????:

$$u(0)=\alpha_1, u(1)=\beta_1$$

, (2)

??????? ???????:

$$\frac{du}{dx}(0)=\alpha_2, \frac{du}{dx}(1)=\beta_2.$$

(3)

?????? (1), (2) ? (1), (3) ???????????? ?????? ??????????? ???????????:

$$J(v)=\int_0^1\left(-\frac{d^2v}{dx^2}+qv-2f\right)vdx=\int_0^1\left(\left(\frac{dv}{dx}\right)^2+qv^2-2fv\right)dx$$

(4)

?????????? ????????? $[0,1]$? ????? $h=\frac{1}{N}$, ??????? ????? $x_i=ih$.

???????????? ??????? ???? ? ???? ???????? ??????????:

$$u_N(x)=\sum_{i=0}^{N}a_i\varphi_i$$

, (5)

??? ???????????? a_i — ???????? ??????? ??????? ? ????? ?????, ? φ_i — ???????? ??????? (????????), ???????? ?? ?????????? $[x_{i-1},x_{i+1}]$?????, ??? $\varphi_i(x_i)=1$, $\varphi_i(x_{i-1})=\varphi_i(x_{i+1})=0$. ??? ???????? ??????? [1, ?. 101]:

$$\varphi_i(x)=\begin{cases}\frac{x-x_{i-1}}{h}, & x\in[x_{i-1},x_i]\\ \frac{x_{i+1}-x}{h}, & x\in[x_i,x_{i+1}]\\ 0, & x\notin[x_{i-1},x_{i+1}]\end{cases}$$

, $i=1,\ldots,N-1$,

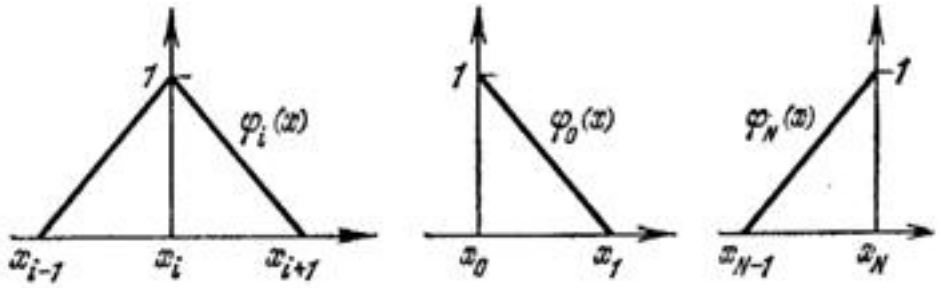
$$\varphi_0(x)=\begin{cases}\frac{x_1-x}{h}, & x\in[x_0,x_1]\\ 0, & x\notin[x_0,x_1]\end{cases}$$

,

$$\varphi_N(x)=\begin{cases}\frac{x-x_{N-1}}{h}, & x\in[x_{N-1},x_N]\\ 0, & x\notin[x_{N-1},x_N]\end{cases}$$

.

?????????????? ????????????? ???????? ???????:



??????? 1. ??????? ???????? ???????

??,???????????
(5) ? (4). ??? ????? ??????? ? ??????? ?????????? ????????? ???:

$$J(u_N) = \sum_{i,j=1}^{N-1} a_i a_j A_{ij} - 2\sum_{i=1}^{N-1} a_i f_i$$

,

???: $f_i = \int_0^1 f(x)\varphi_i(x)dx$, ? $A_{ij} = \int_0^1 \left(\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} + \varphi_i\varphi_j \right) dx$

— ???????? ???????
?????????:

$$\begin{pmatrix}
A_{00} & 0 & 0 & 0 & _ & 0 & 0 & 0 & 0 \\
0 & A_{11} & A_{12} & 0 & _ & 0 & 0 & 0 & 0 \\
0 & A_{21} & A_{22} & A_{23} & _ & 0 & 0 & 0 & 0 \\
_ & _ & _ & _ & _ & _ & _ & _ & _ \\
0 & 0 & 0 & 0 & _ & A_{N-2,N-3} & A_{N-2,N-2} & A_{N-2,N-1} & 0 \\
0 & 0 & 0 & 0 & _ & 0 & A_{N-1,N-2} & A_{N-1,N-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{NN}
\end{pmatrix}$$

(6)

???????? ??????????? ? ??????????? ??????? ????????????? ???????? ????????? ???????????, ??????? ??? ???????? ? ???????????? ?? ? ????. ? ?????????? ???????? ??????? ????:

$$\begin{cases} A_{00}a_0 + A_{01}a_1 = f_0 \\ A_{i,i-1}a_{i-1} + A_{ii}a_i + A_{i,i+1}a_{i+1} = f_i,\ i = \overline{1,N-1} \\ A_{N,N-1}a_{N-1} + A_{NN}a_N = f_N \end{cases} \tag{7}$$

??? ?????? ??????? ? ??????????? ???????? ????????? $u(a) = u(b) = 0,$

??????? ????????? ????? ???:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \_ & 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 & \_ & 0 & 0 & 0 & 0 \\ 0 & A_{21} & A_{22} & A_{23} & \_ & 0 & 0 & 0 & 0 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ 0 & 0 & 0 & 0 & \_ & A_{N-2,N-3} & A_{N-2,N-2} & A_{N-2,N-1} & 0 \\ 0 & 0 & 0 & 0 & \_ & 0 & A_{N-1,N-2} & A_{N-1,N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

?????????????? ??????? ?????????:

$$\begin{cases} a_0 = 0 \\ A_{i,i-1}a_{i-1} + A_{ii}a_i + A_{i,i+1}a_{i+1} = f_i,\ i = \overline{1, N-1} \\ a_N = 0 \end{cases} \quad (9)$$

??? ?????? ??????? ? ????????????? ???????? ????????? $u(0) = u_0, u(1) = u_1,$

??????? ????????? ?????? ???:

$$\begin{pmatrix} u_1 & A_{01} & 0 & 0 & \_ & 0 & 0 & 0 & 0 \\ A_{10} & A_{11} & A_{12} & 0 & \_ & 0 & 0 & 0 & 0 \\ 0 & A_{21} & A_{22} & A_{23} & \_ & 0 & 0 & 0 & 0 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ 0 & 0 & 0 & 0 & \_ & A_{N-2,N-3} & A_{N-2,N-2} & A_{N-2,N-1} & 0 \\ 0 & 0 & 0 & 0 & \_ & 0 & A_{N-1,N-2} & A_{N-1,N-1} & A_{N-1,N} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{N,N-1} & u_1 \end{pmatrix} \quad (10)$$

?????????????? ??????? ?????????:

$$\begin{cases} u_0 + A_{01}a_1 = f_0 \\ A_{i,i-1}a_{i-1} + A_{ii}a_i + A_{i,i+1}a_{i+1} = f_i,\ i = \overline{1, N-1} \\ A_{N,N-1}a_{N-1} + u_1 = f_N \end{cases} \quad (11)$$

? ?????? ??????? ? ??????????? ???????? ????????? $\frac{du}{dx}(0) = \frac{du}{dx}(1) = 0$

???????? ??????? ?? ??????? ???????????? ????????? ???????:

$$u(0) = A_{00}, u(1) = A_{NN}$$

????? ??????? ?????????:

$$\begin{pmatrix} A_{0,0} & A_{0,1} & 0 & 0 & \_ & 0 & 0 & 0 & 0 \\ A_{1,0} & A_{1,1} & A_{1,2} & 0 & \_ & 0 & 0 & 0 & 0 \\ 0 & A_{2,1} & A_{2,2} & A_{2,3} & \_ & 0 & 0 & 0 & 0 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ 0 & 0 & 0 & 0 & \_ & A_{N-2,N-3} & A_{N-2,N-2} & A_{N-2,N-1} & 0 \\ 0 & 0 & 0 & 0 & \_ & 0 & A_{N-1,N-2} & A_{N-1,N-1} & A_{N-1,N} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{N,N-1} & A_{N,N} \end{pmatrix}$$

(12)

? ?????????????? ?? ??????? ?????? ???:

$$\begin{cases} A_{00} + A_{01}a_1 = f_0 \\ A_{i,i-1}a_{i-1} + A_{i,i}a_i + A_{i,i+1}a_{i+1} = f_i, \ i = \overline{1, N-1} \\ A_{N,N-1}a_{N-1} + A_{NN} = f_N \end{cases}$$

(13)

??? ?????? ??????? ? ???????????? ???????? ?????????

$\frac{du}{dx}(0) = u'_0$, $\frac{du}{dx}(1) = u'_1$,

??????? ????????? ????? ??? (

?????????:

$$\begin{cases} A_{00}a_0 + A_{01}a_1 = \tilde{f}_0 \\ A_{i,i-1}a_{i-1} + A_{ii}a_i + A_{i,i+1}a_{i+1} = \tilde{f}_i, \ i = \overline{1, N-1} \\ A_{N,N-1}a_{N-1} + A_{NN}a_N = \tilde{f}_N \end{cases}$$

(14)

???: $\tilde{f}_i = f_i + \frac{1}{2}\left(\beta_2 \cdot \varphi_i(1) - \alpha_2 \cdot \varphi_i(0)\right), \ i = 0\_N$

??? ??? ??????? ????????????? ?????? ????? ??????????????? ???, ?? ????????????? ?????? ?? ??????? ????????.

?????? 1. ????? ??????????? ?? $[0;1]$??????? $u(x)$, ???????????????

????????? $-\frac{d^2u}{dx^2} + u(x) = \sin(x^2)$? ??????? ???????? $u(0) = u(1) = 0$.

? ?????????? ??????? ?????? ??????? ?????? ??? $N = 6$????????:

$h = \frac{1}{6}$, $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{1}{3}, x_3 = \frac{1}{2}, x_4 = \frac{2}{3}, x_5 = \frac{5}{6}, x_6 = 1$.

???????? ???????: $\varphi_0 = \varphi_6 = 0$

$$\varphi_1 = \begin{cases} 0, & x < 0 \\ 6x, & x < \frac{1}{6} \\ 2 - 6x, & x < \frac{1}{3} \\ 0, & otherwise \end{cases}; \qquad \varphi_2 = \begin{cases} 0, & x < \frac{1}{6} \\ -1 + 6x, & x < \frac{1}{3} \\ 3 - 6x, & x < \frac{1}{2} \\ 0, & otherwise \end{cases};$$

$$\varphi_3 = \begin{cases} 0, & x < \frac{1}{3} \\ -2 + 6x, & x < \frac{1}{2} \\ 4 - 6x, & x < \frac{2}{3} \\ 0, & otherwise \end{cases};$$

$$\varphi_4 = \begin{cases} 0, & x < \frac{1}{2} \\ -3 + 2x, & x < \frac{2}{3} \\ 5 - 6x, & x < \frac{5}{6} \\ 0, & otherwise \end{cases}; \qquad \varphi_5 = \begin{cases} 0, & x < \frac{2}{3} \\ -4 + 6x, & x < \frac{5}{6} \\ 6 - 6x, & x < 1 \\ 0, & otherwise \end{cases}.$$

? ??????? ????????? ???????? ??????? ????????? ????? $\frac{218}{3}$,

??????????? ? ???????????? ??????? ?? ????????? $\left(-\frac{215}{6}\right)$. ?????? ????? ???????:

$$f_1 = 0{,}04702;\ f_2 = 0{,}1014;\ f_3 = 0{,}1734;\ f_4 = 0{,}2586;\ f_5 = 0{,}3497.$$

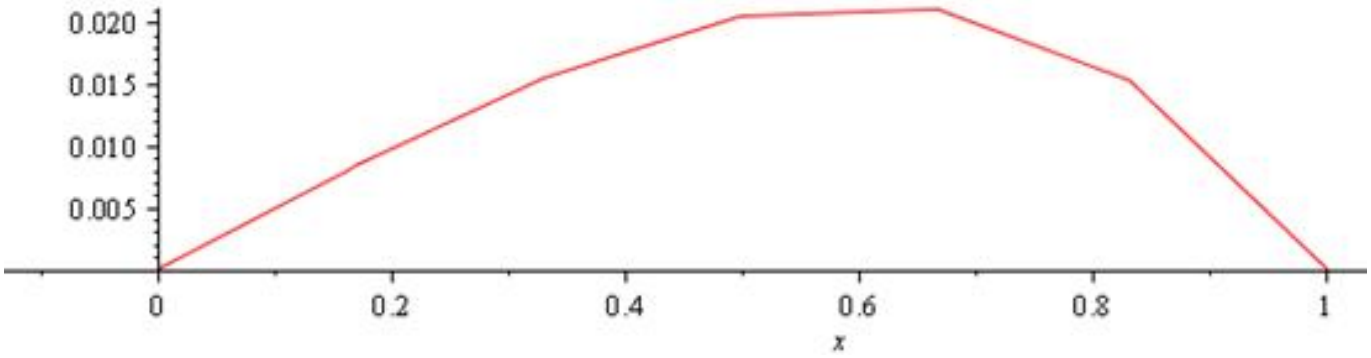
? ?????????? ??????? ???????? ??????? ????????????:

$$a_1 = 0{,}008389;\ a_2 = 0{,}0157;\ a_3 = 0{,}02061;\ a_4 = 0{,}02126;\ a_5 = 0{,}0153.$$

???????????? ??????? ????? ????????? ???:

$$u_6(x) = 0{,}008386 \cdot \varphi_1(x) + 0{,}0157 \cdot \varphi_2(x) + 0{,}02061 \cdot \varphi_3(x) + 0{,}02126 \cdot \varphi_4(x) + 0{,}0153 \cdot \varphi_5(x).$$

?????? ???????????? ??????? ?????????? ?? ??????? 2.



??????? 2. ?????? ??????? ?????? ??????? ? ?????????? ???????? ?????????

?????? 2. ????? ??????? ?????? ??????? ?? $[0;1]$??? ?????????

$$-\frac{d^2u}{dx^2} + u(x) = \sin(x^2)$$

? ??????? ??????? $u(0) = 2,\ u(1) = 3.$????????

N, h, x_i, φ_i ????? ?? ??? ? ??????? 1. ???????? ???????, ??? ????? x_0 ? x_N :

$$\varphi_0 = \begin{cases} 0, & x < 0 \\ 1 - 6x, & x < \frac{1}{6} \\ 0, & otherwise \end{cases} \qquad \varphi_6 = \begin{cases} 0, & x < \frac{5}{6} \\ -5 + 6x, & x < 1 \\ 0, & otherwise \end{cases}$$

; .

? ??????? ????????? $A_{00} = 2, A_{NN} = 3$???????? ?? ??????? ????????? ????? $\frac{218}{3}$, ? ?????????????? ? ?????????????? ???????? ????? $\left(-\frac{215}{6}\right)$.

?????? ????? ???????:

$$f_0 = 0{,}010389;\ f_1 = 0{,}047093;\ f_2 = 0{,}102588;\ f_3 = 0{,}176493;\ f_4 = 0{,}26101;$$
$$f_5 = 0{,}341356;\ f_6 = 0{,}192077.$$

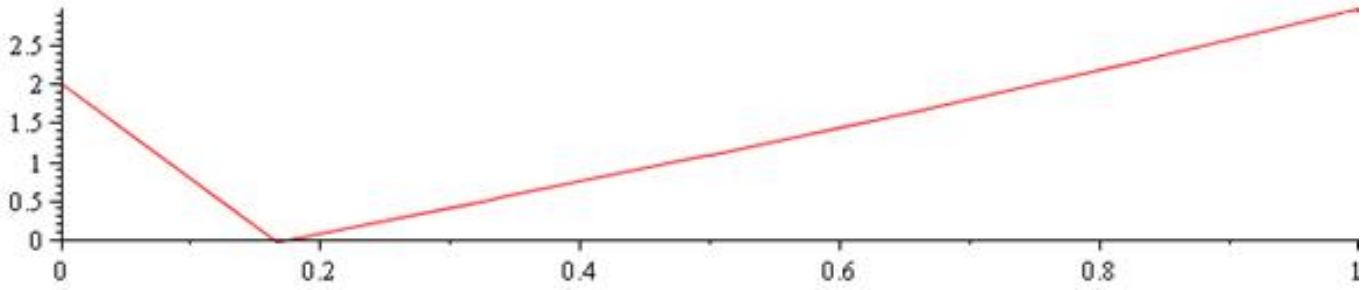
??????? ????????????:

$$a_0 = 2;\ a_1 = -0{,}03327;\ a_2 = 0{,}5221;\ a_3 = 1{,}089;\ a_4 = 1{,}682;\ a_5 = 2{,}314;\ a_6 = 3.$$

??????????? ??????? ????? ????????? ???:

$$u_n(x) = 2 \cdot \varphi_0(x) - 0{,}03327 \cdot \varphi_1(x) + 0{,}5221 \cdot \varphi_2(x) + 1{,}089 \cdot \varphi_3(x) + 1{,}682 \cdot \varphi_4(x) + \\ + 2{,}314 \cdot \varphi_5(x) + 3 \cdot \varphi_6(x).$$

?????? ????????????? ??????? ??????????? ?? ??????? 3.



??????? 3. ?????? ??????? ?????? ??????? ? ????????????? ???????? ?????????

?????? 3. ????? ?? $[0;1]$??????? $u(x)$, ?????????????? ????????? $-\frac{d^2u}{dx^2} + u(x) = \sin(x^2)$? ??????? ???????? $\frac{du}{dx}(0) = 0,\ \frac{du}{dx}(1) = 0$. ????????

$N,\ h,\ x_i,\ \varphi_i$

??? ? ??????? 2.

? ??????? ????????? $A_{00} = \frac{109}{3}, A_{NN} = \frac{109}{3}$??????????? ????????

????? $\frac{218}{3}$, ?????????????? ? ?????????????? $-\left(-\frac{215}{6}\right)$.

?????? ????? ???????:

$$f_0 = 0{,}010389;\ f_1 = 0{,}047093;\ f_2 = 0{,}102588;\ f_3 = 0{,}176493;\ f_4 = 0{,}26101;$$
$$f_5 = 0{,}341356;\ f_6 = 0{,}192077.$$

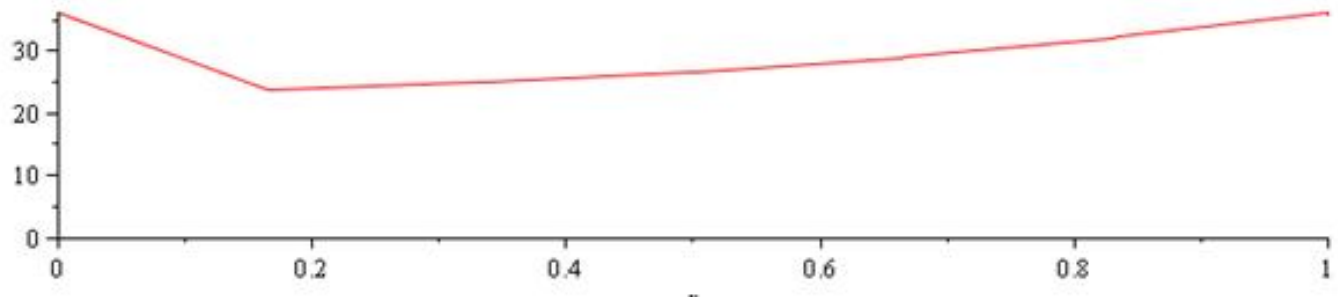
???????? ????????????:

$$a_0 = \frac{109}{3};\ a_1 = 23{,}88;\ a_2 = 24{,}88;\ a_3 = 26{,}57;\ a_4 = 28{,}99;\ a_5 = 32{,}22;\ a_6 = \frac{109}{3}.$$

???????????? ??????? ????? ????????? ???:

$$u_n(x) = \frac{109}{3}\cdot\varphi_0(x) + 23{,}88\cdot\varphi_1(x) + 24{,}88\cdot\varphi_2(x) + 26{,}57\cdot\varphi_3(x) + 28{,}99\cdot\varphi_4(x) +$$
$$+ 32{,}22\cdot\varphi_5(x) + \frac{109}{3}\cdot\varphi_6(x).$$

?????? ???????????? ??????? ?????????? ?? ??????? 4.



??????? 4. ?????? ??????? ?????? ??????? ? ?????????? ???????? ?????????

?????? 4. ????? ?? $[0;1]$??????? $u(x)$, ?????????????? ?????????

$$-\frac{d^2u}{dx^2}+u(x)=\sin(x^2)$$

? ??????? ???????? $\frac{du}{dx}(0)=2,\ \frac{du}{dx}(1)=23.$

???????? $N,\ h,\ x_i,\ \varphi_i$

??? ? ??????? 3.

? ??????? ????????? $A_{00}=\frac{109}{3},A_{NN}=\frac{109}{3}$??????????? ???????? ????? $\frac{218}{3}$, ?????????????? ? ?????????????? $-\left(-\frac{215}{6}\right)$.

?????? ????? ???????:

$$f_0=-859{,}7606;\ f_1=308{,}6828;\ f_2=308{,}7383;\ f_3=308{,}8122;\ f_4=308{,}8967;$$
$$f_5=308{,}9771;\ f_6=-859{,}5789.$$

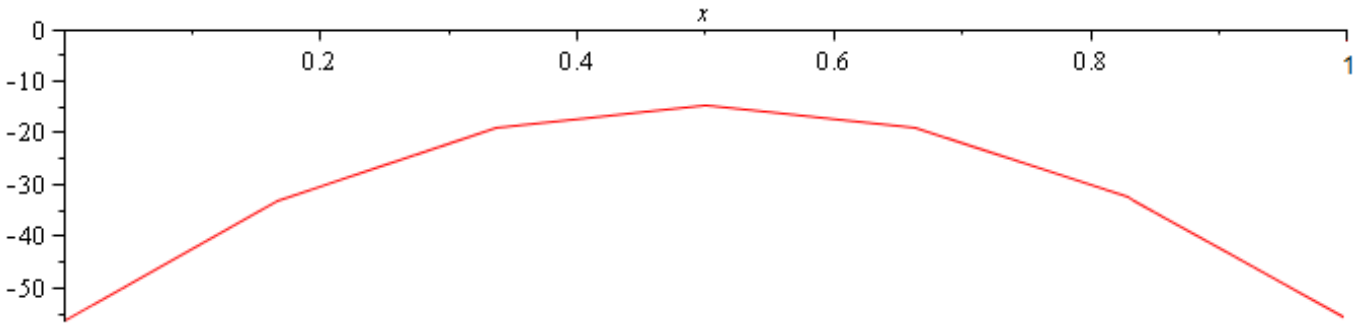
???????? ???????????:

$$a_0=-56{,}14889;\ a_1=-32{,}93904;\ a_2=-19{,}26282;\ a_3=-14{,}74013;\ a_4=-19{,}24681;$$
$$a_5=-32{,}91098;\ a_6=-56{,}116209.$$

??????????? ??????? ????? ????????? ???:

$$u_n(x)=-56{,}14889\cdot\varphi_0(x)-32{,}93904\cdot\varphi_1(x)-19{,}26282\cdot\varphi_2(x)-14{,}74013\cdot\varphi_3(x)-$$
$$-19{,}24681\cdot\varphi_4(x)-32{,}91098\cdot\varphi_5(x)-56{,}116209\cdot\varphi_6(x).$$

?????? ????????????? ??????? ??????????? ?? ??????? 5.



??????? 5. ?????? ??????? ?????? ??????? ? ????????????? ???????? ?????????

?????? ??????????:

1. ?????? ?.?., ??????? ?.?. ???????? ? ???????????-???????? ??????. — ?.: ?????. ??????? ???????? ??????-????????????? ??????????, 1981. — 416 ?.